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A perturbative approach to Snyder space with applications

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Abstract

In this paper, we present a method for studying systems in the modified formulation of quantum mechanics known as Snyder space, proposed by Snyder (1947 *Phys. Rev.* **71** 38–41). Snyder space predicts a modified commutation algebra for position and momentum operators. The method described in this paper introduces operators satisfying the canonical commutation relations and relates them to the position and momentum operators of Snyder space, effectively mapping a problem in Snyder space into a similar problem in standard quantum mechanics. The method is applied to the simple harmonic oscillator (SHO) in one and two dimensions as well as to the one-dimensional infinite square well. The energy spectra are calculated perturbatively for the SHO. We also find an exact spectrum for the one-dimensional infinite square well potential. These results are shown to agree with similar results found elsewhere in the literature.

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1. Introduction

Quantum mechanics is characterized by the appearance of noncommuting operators. In particular, the canonical commutation relations

$$[x_i, p_j] = i\hbar\delta_{ij} \tag{1}$$

$$[x_i, x_j] = 0 \tag{2}$$

$$[p_i, p_j] = 0, \tag{3}$$

are an essential feature of quantum mechanics, from which all the properties of position and momentum can be derived [1]. In considering physical theories beyond standard quantum mechanics, it is therefore natural to consider modifications to the canonical commutation relations. Usually, these modifications manifest themselves in theories of quantum gravity.

For example, a low energy limit of string theory predicts the deformation of the Heisenberg algebra given by

$$[x_i, p_j] = \mathrm{i}\hbar\delta_{ij} \tag{4}$$

$$[x_i, x_j] = \mathrm{i}\theta_{ij} \tag{5}$$

$$[p_i, p_j] = 0, \tag{6}$$

where θ_{ij} is a real, antisymmetric matrix that characterizes the noncommutativity among spatial coordinates [2, 3].

Historically, the first formulation of quantum mechanics on a noncommutative manifold, known as Snyder space, was proposed by Hartland Snyder in 1947 [4, 5]. Snyder introduced a fundamental length a with the hope of removing divergence problems from quantum field theory. A natural consequence of this introduction is the modification of the canonical commutation relations to

$$[x_i, p_j] = \mathrm{i}\hbar(\delta_{ij} + \alpha p_i p_j) \tag{7}$$

$$[x_i, x_j] = \mathrm{i}\hbar\alpha\epsilon_{ijk}L_k \tag{8}$$

$$[p_i, p_j] = 0, \tag{9}$$

where $\alpha = a^2/\hbar^2$ and L_k is the angular momentum which is defined in the usual way:

$$L_k = \epsilon_{ijk}x_i p_j. \tag{10}$$

While the divergences of quantum field theory were ultimately dealt with by renormalization, the concept of quantized spacetime remains current as a possible approach when quantizing gravity. Besides its historical significance, Snyder space stands out as a particular quantization of space that preserves Lorentz invariance.

Commutation relations very similar to those proposed by Snyder were derived independently by Kempf *et al* [6, 7], and are known as the minimal length uncertainty relations. This commutation algebra was motivated by possible application to the description of nonpointlike particles, such as strings. The remarkable similarity between the minimal length uncertainty relations and the Snyder commutation relations further justifies reconsidering the consequences of Snyder's formulations.

Another modified commutation algebra is known as dynamical quantization, which was proposed in the early 1980s as an effective method of describing high-energy interactions. In the case of the one-dimensional free particle, the commutation algebra of dynamical quantization is the same as that of Snyder space [8, 9]. Further connections with other approaches to quantum gravity, such as noncommutative quantum mechanics, are explored in [10]. The fact that Snyder space is Lorentz invariant also singles it out among noncommutative theories [11].

In this paper, we discuss a method for approaching problems in Snyder space. The method is analogous to an approach for solving problems in the noncommutative geometry of equations (4)–(6) [3]. The idea is to relate operators satisfying the canonical commutations relations to the physical operators satisfying a modified commutation algebra. Effectively, this maps problems from Snyder space to analogous problems in standard quantum mechanics. This technique is useful because of the existence of numerous techniques for solving systems in standard quantum mechanics. In particular, we will see that the effects of quantized spacetime can be represented as a small additional term in the Hamiltonian of the analogous system in quantum mechanics, so that the perturbation theory is likely to be useful.

2. Operator transformations

The modified (noncanonical) commutation relations presented in equations (7)–(9) are restated as relations between the physical Snyder operators x_S and p_S,

$$[x_{Si}, p_{Sj}] = \mathrm{i}\hbar(\delta_{ij} + \alpha p_{Si} p_{Sj}) \tag{11}$$

$$[x_{Si}, x_{Sj}] = \mathrm{i}\hbar\alpha\epsilon_{ijk} L_{Sk} \tag{12}$$

$$[p_{Si}, p_{Sj}] = 0, \tag{13}$$

where the subscript S, not an index, refers to 'Snyder'. Since x_{Si} and p_{Si} represent physical observables in Snyder space, they are assumed to be Hermitian.

On the other hand, we denote operators that satisfy the canonical commutation relations without the S subscript,

$$[x_i, p_j] = \mathrm{i}\hbar\delta_{ij} \tag{14}$$

$$[x_i, x_j] = 0 \tag{15}$$

$$[p_i, p_j] = 0. \tag{16}$$

Note that these commutation relations can be recovered from those in equations (11)–(13) in the limit that α approaches 0. Since x_i and p_i do not represent physical variables, they are not necessarily Hermitian. However, since our goal is to map problems into standard quantum mechanics by means of these operators, we add the requirement that they be Hermitian as well.

We wish to find an expression for x_S and p_S in terms of operators satisfying the canonical commutation relations, x and p. We refer to such a relation as an operator transformation, since it allows us to effectively 'transform' operator expressions from Snyder space into an analogous expression in standard quantum mechanics.

First, we consider the one-dimensional case. Using the result on the commutator of two functions of two operators in [12], the commutation relation

$$[x_S, p_S] = \mathrm{i}\hbar\left(1 + \alpha p_S^2\right) \tag{17}$$

can be written as the differential equation

$$\sum_{k=1}^{\infty} \frac{(-\mathrm{i}\hbar)^k}{k!} \left(\frac{\partial^k p_S}{\partial x^k} \frac{\partial^k x_S}{\partial p^k} - \frac{\partial^k x_S}{\partial x^k} \frac{\partial^k p_S}{\partial p^k} \right) = \mathrm{i}\hbar\left(1 + \alpha p_S^2\right), \tag{18}$$

where the operators x_S and p_S are functions of the operators x and p. The use of equation (18) automatically imposes the canonical behaviour of x and p. Since equation (18) is a single equation for two unknowns, there is some freedom in our choice of x_S and p_S. By making a judicious choice for one of x_S or p_S, equation (18) can be greatly simplified.

If we make the identification

$$p_S = p, \tag{19}$$

then equation (18) becomes

$$(\mathrm{i}\hbar)\frac{\partial x_S}{\partial x} = \mathrm{i}\hbar(1 + \alpha p^2) \tag{20}$$

which has the solution

$$x_S = (1 + \alpha p^2)x + f(p) \tag{21}$$

where $f(p)$ is an arbitrary function of p that can be determined by the hermiticity requirement on x_S. In particular, $x_S^\dagger = x_S$ gives

$$x(1+\alpha p^2) + f(p)^\dagger = (1+\alpha p^2)x + f(p) \tag{22}$$

or equivalently

$$f(p) - f(p)^\dagger = \alpha(xp^2 - p^2x) = 2\mathrm{i}\hbar\alpha p, \tag{23}$$

where we have assumed that x and p are Hermitian. This can be solved by assuming that $f(p)$ is anti-Hermitian, $f(p) = \mathrm{i}\hbar\alpha p$. Putting it all together, we have

$$\begin{aligned} x_S &= (1+\alpha p^2)x + \mathrm{i}\hbar\alpha p = x + \tfrac{1}{2}\alpha(p^2x + xp^2) = x + \alpha pxp \\ p_S &= p. \end{aligned} \tag{24}$$

Similarly, if we make the identification

$$x_S = x \tag{25}$$

then equation (18) becomes

$$\mathrm{i}\hbar\frac{\partial p_S}{\partial p} = \mathrm{i}\hbar\left(1+\alpha p_S^2\right). \tag{26}$$

This differential equation is satisfied by $p_S = \frac{1}{\sqrt{\alpha}}\tan(\sqrt{\alpha}p)$. The whole transformation is then

$$x_S = x \qquad p_S = \frac{\tan(\sqrt{\alpha}p)}{\sqrt{\alpha}}. \tag{27}$$

In order for the transformation to be invertible, we restrict the range of p, such that $-\pi/2 \leqslant \sqrt{\alpha}p \leqslant \pi/2$. Using this transformation, the quantization effects are manifest by truncating the range of p to a finite interval. This issue is further discussed in the appendix. Analogously, in standard quantum mechanics, truncating the range of the position operator by means of the infinite square well results in a quantization of the energy and the momentum [10].

Following a similar approach in two dimensions, we obtain a transformation, leaving momentum unaffected

$$\begin{aligned} p_{Sx} &= p_x \\ p_{Sy} &= p_y \\ x_S &= x + \alpha\left(p_x x p_x + \tfrac{1}{2}p_x(yp_y + p_y y)\right) \\ y_S &= y + \alpha\left(p_y y p_y + \tfrac{1}{2}p_y(xp_x + p_x x)\right) \end{aligned} \tag{28}$$

and transformations leaving just one of the coordinates unaffected

$$\begin{aligned} x_S &= x \\ p_{Sx} &= \frac{\tan(\sqrt{\alpha}p_x)}{\sqrt{\alpha}} \\ y_S &= -y\cos(\sqrt{\alpha}p_x) + \cot(\sqrt{\alpha}p_y)\sin(\sqrt{\alpha}p_x)x \\ p_{Sy} &= \frac{\cot(\sqrt{\alpha}p_y)}{\sqrt{\alpha}\cos(\sqrt{\alpha}p_x)}, \end{aligned} \tag{29}$$

where a similar transformation is found by letting $y_S = y$ with appropriate expressions for the other operators.

In the one-dimensional case we found that the commutation algebra provided one equation to be satisfied by two operators, which allowed us to arbitrarily specify one of the operators. In d dimensions, there are $2d$ operators that must satisfy $\binom{2d}{2}$ equations. Therefore, in higher dimensions there are more equations than unknowns and the system is overdetermined. One can always find at least one transformation in arbitrary dimensions. It is given by

$$x_{Si} = x_i + \tfrac{1}{2}\alpha p_i(x_j p_j + p_j x_j) \tag{30}$$

$$p_{Si} = p_i, \tag{31}$$

where we note the use of the Einstein summation convention. One can show that this transformation satisfies the commutation algebra in arbitrary dimensions.

The applicability of this technique can be found by considering the functional form of a Hamiltonian H under a transformation described in this section. All of the transformations described in this section can be written as

$$\vec{x}_S = \vec{x} + \vec{X}(\alpha, \vec{x}, \vec{p}) \tag{32}$$

$$\vec{p}_S = \vec{p} + \vec{P}(\alpha, \vec{x}, \vec{p}), \tag{33}$$

where $\vec{X}$ and $\vec{P}$ depend on the choice of transformation and vanish in the limit that $\alpha \to 0$. Given such a transformation, the Hamiltonian becomes

$$H(\vec{x}_S, \vec{p}_S) = H(\vec{x} + \vec{X}(\alpha, \vec{x}, \vec{p}), \vec{p} + \vec{P}(\alpha, \vec{x}, \vec{p})). \tag{34}$$

If the Hamiltonian is an analytic function of its arguments it can be expanded as a Taylor series as

$$H(\vec{x}_S, \vec{p}_S) = H(\vec{x}, \vec{p}) + \alpha V_1(\vec{x}, \vec{p}) + \alpha^2 V_2(\vec{x}, \vec{p}) + \cdots, \tag{35}$$

which is a Hamiltonian of the same system in standard quantum mechanics with small correction terms. The system can then be tackled either exactly or through techniques of perturbation theory.

3. Application to the simple harmonic oscillator

As an application of the transformations presented in section 2, we study the simple harmonic oscillator in Snyder space. In one dimension, using the transformation of equation (24) the Hamiltonian is

$$H = \frac{p_S^2}{2m} + \frac{1}{2}m\omega^2 x_S^2 \tag{36}$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}\alpha m\omega^2(2x^2p^2 - 4i\hbar xp - \hbar^2) + \frac{1}{2}\alpha^2 m\omega^2(x^2p^4 - 4i\hbar xp^3 - 2\hbar^2 p^2). \tag{37}$$

This is clearly a simple harmonic oscillator with small perturbation terms proportional to α and α^2. Furthermore, the expression for the Hamiltonian does not include references to the Snyder space operators, although there are terms that include the fundamental length, as expected. Since the operators x and p are Hermitian, the Hamiltonian in equation (37) is a valid Hamiltonian in standard quantum mechanics and standard solution techniques apply.

One can introduce the ladder operators a and $a^\dagger$ in the usual way

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\mathrm{i}}{m\omega}p\right) \tag{38}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\mathrm{i}}{m\omega}p\right), \tag{39}$$

and express the Hamiltonian in terms of a and $a^\dagger$ to obtain

$$\begin{aligned} H = \frac{p_S^2}{2m} + \frac{1}{2}m\omega^2 x_S^2 &= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + \frac{1}{4}\hbar^2\omega^2\alpha m\left(2a^{\dagger 2}a^2 + 4a^\dagger a + 1 - a^{\dagger 4} - a^4\right) \\ &+ \frac{1}{16}\hbar^3\omega^3\alpha^2 m^2\left(7 + 26a^\dagger a + 18a^{\dagger 2}a^2 + 4a^{\dagger 3}a^3\right) \\ &+ \frac{1}{16}\hbar^3\omega^3\alpha^2 m^2\left(-7a^2 - 7a^{\dagger 2} - 5a^{\dagger 4} - 5a^4 - 4a^\dagger a^3 - 4a^{\dagger 3}a\right) \\ &+ \frac{1}{16}\hbar^3\omega^3\alpha^2 m^2\left(a^6 + a^{\dagger 6} - 2a^{\dagger 5}a - 2a^\dagger a^5 - a^{\dagger 2}a^4 - a^{\dagger 4}a^2\right). \end{aligned} \tag{40}$$

We note that in the above expression, terms that are self-adjoint are listed first among the terms corresponding to a given power of the fundamental length. They are the only ones contributing in first-order perturbation. Terms that are not self-adjoint are then listed in increasing powers of the ladder operators. Inspection reveals that this Hamiltonian is self-adjoint.

The Hamiltonian given above, expressed both in terms of position and momentum or in terms of raising and lowering operators, is exact. No approximations have been made to this point. By studying the form of equation (37), we can see that the effect of quantizing spacetime is to introduce a perturbation that includes terms containing position and momentum. This can be interpreted as a velocity-dependent and position-dependent force.

In one dimension, it may be more useful to apply the transformation given in equation (27). In this case, for any velocity-independent potential, the result of the transformation will be to modify the kinetic energy T to

$$T = \frac{p_S^2}{2m} = \frac{\alpha}{2m}\tan^2\left(\frac{p}{\sqrt{\alpha}}\right), \tag{41}$$

while leaving the potential energy term V unchanged

$$V(x_S) = V(x). \tag{42}$$

In this case, the perturbation is always a velocity-dependent and position-independent perturbation.

Once an exact form of the Hamiltonian is found, it can be solved by applying techniques from perturbation theory. Using perturbation theory formulae worked out to fourth-order in α [10, 13], the corrected energy spectrum of equation (40) is

$$E_n = \hbar\omega\left(\left(n + \frac{1}{2}\right)\left(1 + \frac{1}{8}\beta^2 - \frac{1}{128}\beta^4\right) + \frac{1}{4}\beta(2n^2 + 2n + 1)\right), \tag{43}$$

where $\beta = \alpha m\omega\hbar$ is the dimensionless perturbation expansion parameter. Repeating the procedure using the transformation given in equation (27) instead of equation (24) results in the same energy spectrum.

The same process can be applied to the two-dimensional isotropic harmonic oscillator; however, the situation is complicated by the fact that the energy spectrum of the two-dimensional harmonic oscillator is degenerate. The angular momentum operator, L_{Sz}, can be

shown to commute with any radially symmetric Hamiltonian [10]. Therefore, in this case we can find a basis for which the perturbation series converges by finding the angular momentum eigenstates within each degenerate set of states.

In the two-dimensional case, we choose the transformation in equation (28). The two-dimensional Hamiltonian is then

$$H = H_0 + V_\alpha + V_{\alpha^2} \tag{44}$$

$$H_0 = \omega\hbar(a^\dagger a + b^\dagger b + 1) \tag{45}$$

$$\begin{aligned} V_\alpha = {} & \tfrac{1}{4}\omega^2 h^2 \alpha m\left(2b^{\dagger 2}b^2 + 4b^\dagger b + 2 + 4a^\dagger a + 2a^{\dagger 2}a^2\right) \\ & + \tfrac{1}{4}\omega^2 h^2 \alpha m\left(-b^{\dagger 4} - 2a^2 b^2 - b^4 + 2a^2 b^{\dagger 2}\right) \\ & + \tfrac{1}{4}\omega^2 h^2 \alpha m\left(2a^{\dagger 2}b^2 - 2a^{\dagger 2}b^{\dagger 2} - a^{\dagger 4} - a^4\right) \end{aligned} \tag{46}$$

$$\begin{aligned} V_{\alpha^2} = {} & \tfrac{1}{16}\omega^3\hbar^3 m^2\alpha^2\left(18 + 20b^{\dagger 2}b^2 + 16a^\dagger a b^\dagger b + 34b^\dagger b + 4a^{\dagger 3}a^3\right) \\ & + \tfrac{1}{16}\omega^3\hbar^3 m^2\alpha^2\left(4a^{\dagger 2}a^2 b^\dagger b + 4b^{\dagger 3}b^3 + 4a^\dagger a b^{\dagger 2}b^2 + 34a^\dagger a + 20a^{\dagger 2}a^2\right) \\ & + \tfrac{1}{16}\omega^3\hbar^3 m^2\alpha^2\left(-9a^2 - 9b^2 - 9a^{\dagger 2} - 9b^{\dagger 2}\right) \\ & + \tfrac{1}{16}\omega^3\hbar^3 m^2\alpha^2\left(-6a^{\dagger 4} - 4a^{\dagger 2}b^\dagger b - 4b^\dagger b^3 + 12a^{\dagger 2}b^2 - 12a^{\dagger 2}b^{\dagger 2}\right) \\ & + \tfrac{1}{16}\omega^3\hbar^3 m^2\alpha^2\left(12a^2 b^{\dagger 2} - 4a^\dagger a b^2 - 4b^{\dagger 3}b - 4a^\dagger a^3 - 6a^4 - 6b^{\dagger 4}\right) \\ & + \tfrac{1}{16}\omega^3\hbar^3 m^2\alpha^2\left(-6b^4 - 4a^2 b^\dagger b - 4a^{\dagger 3}a - 4a^\dagger a b^{\dagger 2} - 12a^2 b^2\right) \\ & + \tfrac{1}{16}\omega^3\hbar^3 m^2\alpha^2\left(-2a^{\dagger 4}b^\dagger b + 4a^2 b^{\dagger 3}b - a^{\dagger 4}b^2 + 3a^{\dagger 4}b^{\dagger 2} + 3a^4 b^2 - a^4 b^{\dagger 2}\right) \\ & + \tfrac{1}{16}\omega^3\hbar^3 m^2\alpha^2\left(4a^{\dagger 3}a b^2 - 4a^{\dagger 3}a b^{\dagger 2} - 4a^\dagger a^3 b^2 + 4a^\dagger a^3 b^{\dagger 2} - a^{\dagger 4}a^2\right) \\ & + \tfrac{1}{16}\omega^3\hbar^3 m^2\alpha^2\left(-a^{\dagger 2}a^4 - 2b^{\dagger 5}b - b^{\dagger 4}b^2 - 2a^{\dagger 2}a^2 b^2 - 2a^2 b^{\dagger 2}b^2 + a^{\dagger 6}\right) \\ & + \tfrac{1}{16}\omega^3\hbar^3 m^2\alpha^2\left(-2a^{\dagger 5}a - b^{\dagger 2}b^4 - 2a^4 b^\dagger b - 4a^{\dagger 2}b^{\dagger 3}b - 2a^\dagger a b^{\dagger 4} - a^{\dagger 2}b^4\right) \\ & + \tfrac{1}{16}\omega^3\hbar^3 m^2\alpha^2\left(-2a^{\dagger 2}b^{\dagger 2}b^2 + b^{\dagger 6} + a^6 + 4a^{\dagger 2}b^\dagger b^3 - 2a^\dagger a b^4 + 3a^2 b^4\right) \\ & + \tfrac{1}{16}\omega^3\hbar^3 m^2\alpha^2\left(-2a^{\dagger 2}a^2 b^{\dagger 2} + b^6 + 3a^{\dagger 2}b^{\dagger 4} - 2b^\dagger b^5\right) \\ & + \tfrac{1}{16}\omega^3\hbar^3 m^2\alpha^2\left(-4a^2 b^\dagger b^3 - a^2 b^{\dagger 4} - 2a^\dagger a^5\right) \end{aligned} \tag{47}$$

where a and $a^\dagger$ are defined in equations (38) and (39) with p replaced by p_x and b and $b^\dagger$ are the corresponding ladder operators for the second dimension,

$$b = \sqrt{\frac{m\omega}{2\hbar}}\left(y + \frac{\mathrm{i}}{m\omega}p_y\right) \tag{48}$$

$$b^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(y - \frac{\mathrm{i}}{m\omega}p_y\right). \tag{49}$$

The terms are ordered following same ordering as in equation (40).

A brief comparison of the perturbation terms in one and two dimensions shows a significant increase in complexity. The derivation of the Hamiltonian in equation (44) involves a considerable amount of algebra. Furthermore, the evaluation of formulae from perturbation theory is also quite involved. We derived the exact form of the Hamiltonian and evaluated the perturbation formulae using algorithms developed for that purpose in Maple [10].

The second-order perturbation corrected energy values are given by

$$E_{n,l} = \hbar\omega\left(n + 1 + \frac{\beta}{2}((n+1)^2 - l^2) + \frac{\beta^2}{8}(n+1)\right), \tag{50}$$

where l is the angular momentum quantum number, which takes on integer values between n and $-n$ in intervals of 2. Repeating the proceduring using the transformation in equation (29) results in the same energy spectrum.

This analysis represents a proof of principle that the method of operator transformations can be applied to find energy spectra in Snyder space. The simple harmonic oscillator was chosen because it has the advantage that it has an exact solution in standard quantum mechanics. In particular, the solution can be found by algebraic methods, so it is relatively straightforward to apply methods from perturbation theory. Even in this case, the perturbation terms in the Snyder simple harmonic oscillator are relatively complicated. We evaluated the perturbation spectra by constructing algorithms in Maple and applying them to the specified correction terms [10]. For more complex systems, it is reasonable to expect that the perturbation terms could be more complicated. However, they can be constructed in principle and, although the calculations may be more involved, evaluated by application of the algorithms.

4. Application to the infinite square well

As another application of the transformations presented in section 2, we study the particle in a one-dimensional infinite square well in Snyder space. The Hamiltonian is

$$H = \frac{p_S^2}{2m} + V(x_S), \tag{51}$$

where $V(x_S) = 0$ for $0 \leqslant x_S \leqslant L$ and infinite elsewhere.

In order to apply the techniques demonstrated in the previous section, we cannot use the transformation in equation (24). If we did apply that transformation, our next step would create a perturbation series in α; however, since the potential energy function is not analytic, no such series exists for this transformation. Therefore, we choose the transformation in equation (27) as a tool for analysing this system. In this case, the Hamiltonian becomes

$$H = \frac{\tan^2(\sqrt{\alpha}\,p)}{2m\alpha} = \frac{p^2}{2m} + \alpha\frac{p^4}{3m} + \cdots. \tag{52}$$

Using this transformation, only even powers of p appear. Therefore, this Hamiltonian has common eigenstates with $H_0 = p^2/2m$. Hence, we can sum the perturbation series exactly by noting that

$$H = \frac{\tan^2(\sqrt{2m\alpha H_0})}{2m\alpha}. \tag{53}$$

Therefore, the energy eigenvalue spectrum is given by

$$E_n = \frac{\tan^2(\sqrt{2m\alpha E_{0n}})}{2m\alpha}, \tag{54}$$

where E_{0n} are the eigenvalues of the unperturbed infinite square well in standard quantum mechanics. It is well known for a one-dimensional infinite square well that $E_{0n} = \frac{n^2\hbar^2\pi^2}{2mL^2}$. For convenience, we introduce the dimensionless parameter λ

$$\lambda = \frac{L}{a} \tag{55}$$

where a is the fundamental length in Snyder space. Recalling that $\alpha = \frac{a^2}{\hbar^2}$, we find that the energy spectrum in Snyder space is

$$E_n = \frac{\tan^2(n\pi/\lambda)}{2m\alpha}. \tag{56}$$

In section 2, it was shown that using the transformation of equation (27) requires one to truncate the range of the operator p. We consider the inverse transformation

$$\begin{aligned} x &= x_S \\ p &= \frac{\arctan(\sqrt{\alpha}\, p_S) + N\pi}{\sqrt{\alpha}}, \end{aligned} \tag{57}$$

with N integer and we select $N = 0$. Therefore, the quantum number n in equation (56) does not range over all positive integers in the Snyder space spectrum, but has the range $n = 1, 2, \ldots, n_{\max}$ where

$$n_{\max} = \lfloor \lambda/2 \rfloor, \tag{58}$$

where $\lfloor x \rfloor$ denotes the largest integer less than x. Thus, the infinite square well has only a finite number of energy levels, in contrast to the standard quantum mechanical result.

5. Agreement with results in the literature

The simple harmonic oscillator in Snyder space has an exact solution. As mentioned in section 1, the Snyder space commutation algebra is essentially the minimal length uncertainty relations. The simple harmonic oscillator eigenvalue problem has been solved in arbitrary dimensions with the minimal length uncertainty relations by Chang *et al* [14]. Their result indicates that the exact energy spectrum of the Snyder space SHO in d dimensions is given by

$$E_{nl} = \hbar\omega \left(\left(n + \frac{d}{2} \right) \sqrt{1 + \left(\frac{\beta}{2} \right)^2} + \frac{\beta}{2} \left(n + \frac{d}{2} \right)^2 - \frac{\beta}{2} \left(l^2 + \frac{d^2}{4} \right) + \frac{\beta d}{4} \right). \tag{59}$$

Expanding this result as a power series in β recovers the perturbative results derived in section 3 of this paper. A more comprehensive discussion of the exact solution to the SHO in Snyder space will be given elsewhere [15].

The infinite square well is essentially a free particle constrained to move in a finite region. As mentioned in section 2, the Snyder space commutation algebra is equivalent to the dynamical quantization algebra of a free particle [8]. The energy spectrum of a particle in an infinite rigid box has been solved using the commutation relations of dynamical quantization. The results agree with the ones presented in section 4 of this paper [9].

6. Conclusions

This paper presents a proof of principle for the method of operator transformation as an approach for solving energy eigenvalue problems in Snyder space. We have seen that there are several transformations that preserve the Snyder space commutation algebra while relating it to the canonical algebra of quantum mechanics. It is possible that there are others besides the ones given in this paper, although they are likely to be more complicated.

We have applied the techniques of operator transformation to the simple harmonic oscillator in one and two dimensions. The resulting Hamiltonian is also a simple harmonic oscillator with perturbation terms proportional to powers of the fundamental length of Snyder

space. The perturbation terms include factors of momentum, and, therefore, can be interpreted as velocity-dependent forces. The spectrum is independent of the particular transformation chosen.

Application of perturbation theory results in a modified energy spectrum. It is interesting to note that the effects of introducing a minimal length modify the structure of the energy spectrum. Chang *et al* have commented on the possible use of this modified structure as a potential experimental verification of the minimal length hypothesis [14].

We have also applied the technique to the one-dimensional infinite square well. We noted that since the potential energy function is not analytic, not all transformations are practical. We applied a single transformation and found that the resulting Hamiltonian had common eigenstates with the Hamiltonian of standard quantum mechanics, which allowed us to calculate the energy spectrum exactly. This system also resulted in a spectrum very different from that of standard quantum mechanics, possessing only a finite number of energy levels.

The algebraic approach we have illustrated here is actually not only applicable to systems in Snyder space, but also to virtually any modified commutation algebra that reduces to the canonical commutation relations in an appropriate limit.

As further validation of this approach, we have seen that the operator transformation method produces results that agree with results currently available in the literature but obtained using different methods.

Appendix: Why the values of the momentum must be truncated

As we describe in section 2 in the one-dimensional case, there exists a transformation given by

$$x_S = x \tag{A.1}$$

$$p_S = \frac{\tan(\sqrt{\alpha}\,p)}{\sqrt{\alpha}}. \tag{A.2}$$

When using this transformation, it is also necessary to truncate the range of the operator p to make the transformation invertible.

To fully understand the need for this restriction, one must look more closely at the nature of Snyder space [4]. One-dimensional Snyder space is defined by operators acting on the circle defined by

$$\eta^2 = \eta_1^2 + \eta_4^2. \tag{A.3}$$

By making a change of variables to polar coordinates, with polar angle ϕ, the position and momentum operators, x_S and p_S, as defined by Snyder are given by (recalling that $\alpha = a^2/\hbar^2$)

$$x_s = \mathrm{i}a\frac{\mathrm{d}}{\mathrm{d}\phi} \tag{A.4}$$

$$p_S = \frac{\hbar}{a}\tan(\phi). \tag{A.5}$$

By equating

$$p = \frac{\phi}{\sqrt{\alpha}} \tag{A.6}$$

$$x = \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}p} \tag{A.7}$$

we recover the transformation in question.

Snyder quantizes spacetime by imposing the eigenfunctions of the position operator to be single-valued, i.e. to repeat its values as ϕ increases by 2π. Using this condition, one can see that x_S has eigenvalues ma where m is any integer. As a matter of fact, for the projective coordinates of this de Sitter space, it is the ratio η_4/η_1 that really matters, corresponding to a periodicity of π for observable phenomena. Likewise, we require that eigenfunctions of the Hamiltonians that we consider be single-valued, or periodic functions of ϕ.

Using the relation between ϕ and p given above and the periodic requirement on ϕ, we require the same periodic requirement on p. Because of this periodicity, all results can be found by studying a single period and no information is lost restricting the variable to one period, thus the truncation.

In connection with the infinite square well from section 4, we note that since the modified Hamiltonian contains only even powers of the momentum, it has the same eigenstates as the corresponding system in standard quantum mechanics. Therefore, the position representation eigenstates are given by

$$\langle x|\psi\rangle = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \tag{A.8}$$

for $0 \leqslant x \leqslant L$ and 0 elsewhere. And the momentum representation eigenstates are given by the Fourier transform

$$\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_0^L \mathrm{d}x\, \mathrm{e}^{\mathrm{i}px/\hbar}\langle x|\psi\rangle. \tag{A.9}$$

In Snyder space, the quantization of space follows from requiring p to be periodic. However, $\langle p|\psi\rangle$ as given in equation (A.9) is not explicitly periodic. There are no remaining parameters to be fixed in order to impose periodicity on $\langle p|\psi\rangle$. The resulting expression cannot be considered a result characteristic of Snyder space until the periodicity condition is met. This is achieved by imposing the condition directly by defining

$$\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_0^L \mathrm{d}x\, \mathrm{e}^{\mathrm{i}px/\hbar}\langle x|\psi\rangle \tag{A.10}$$

for $-\pi/2 \leqslant \sqrt{\alpha}p \leqslant \pi/2$ with other values defined recursively by $\psi(p + \pi\sqrt{\alpha}) = \psi(p)$. This is equivalent to simply restricting the value of p to the corresponding interval.

The particular interval is symmetric around $p = 0$ and ensures that the Snyder space result reduces to the standard quantum-mechanical result in the low-energy limit with a non-zero energy for the ground state.

If it were possible to satisfy the periodic boundary condition by appropriately choosing constants introduced by solving the differential equations, then standard quantum mechanics would not explain how to choose these constants. It is reasonable to expect that, in mapping problems from Snyder space to standard quantum mechanics, the solutions in standard quantum mechanics contain excess information that should be discarded. This excess information is a consequence of standard quantum mechanics being calculated on a continuous space, while Snyder space uses discrete positions.

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